Problem 1

1) 当x, y同为偶数或同为奇数, x+y为偶数f(x+y)=1, f(x)·f(y)=1, f(x+y)=f(x)·f(y)

当x, y为一个奇数与一个偶数, x+y为奇数, f(x+y)=-1, f(x)·f(y)=-1, f(x+y)=f(x)·f(y)

说明f是群G1到G2的同态, 但f既不是单射也不是满射

则既不是单同态, 也不是满同态, 更不是同构. f(G1) = {1, -1}

2) f(x+y) = cos(x+y) + i·sin(x+y), f(x)·f(y) = (cosx+i·sinx)(cosy+i·siny)

= (cosxcosy-sinxsiny)+i·(cosxsiny+sinxcosy) = cos(x+y) + i·sin(x+y) = f(x+y)

f是群G1到G2的同态, 又f(x)在R上的周期为2π, f在Z上是单射, f是单同态

对f(x)∈A有|f(x)|² = cos²x + sin²x恒成立, f在Z上不是满射, f不是满同态

f不是同构, f(G1) = {cosx + i·sinx | x∈Z}

Problem 2

设G是生成元为a的循环群, 则G=<a>, 任取x, y∈G有m, n∈Z使x=a^m, y=a^n

x·y = a^m·a^n = a^(m+n) = a^(n+m) = a^n·a^m = y·x, G满足交换性, G是阿贝尔群

阿贝尔群不一定是循环群, 如克莱因四元群是阿贝尔群却不是循环群

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | e | a | b | c |
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

Problem 3

设G1是生成元为a的循环群, 则G1=<a>, 任取x∈G1有m∈Z使x=a^m

令y=f(x)=f(a^m)∈f(G1), y=f(a)f(a)……f(a) (m个f(a)), 则f(a)是f(G1)的生成元

即f(G1)=<f(a)>, 可见f(G1)也是循环群

Problem 4

1) 小于或等于15且与15互素的数是1, 2, 4, 7, 8, 11, 13, 14

则G的所有生成元为a, a^2, a^4, a^7, a^8, a^11, a^13, a^14

2) 15的因子有1, 3, 5, 15则G的所有子群为

<a>=G, <a^3>={e, a^3, a^6, a^9, a^12}, <a^5>={e, a^5, a^10}, <a^15>={e}

Problem 5

取任意三阶群G, |G|=3, 取a∈G且a≠e, 构造以a为生成元的循环群<a>, <a>∈G

|<a>| | |G|=3, 又a≠e, |<a>|≠1, |<a>|=3=|G|, G={e, a, a²}是三阶循环群

(实际上, 同理可得所有素数阶群都是循环群)

Problem 6

任取a, b∈G (a≠e, b≠e), 有a\*a=e, b\*b=e, a\*b\*a\*b = (a\*b)\*(a\*b) = e

则b\*a = b\*a\*(a\*b\*a\*b) = b\*(a\*a)\*b\*a\*b = (b\*b)\*a\*b = a\*b, <G, \*>是阿贝尔群

Problem 7

G为阿贝尔群, 任取a, b∈G有ab=ba, (ab)²=(ab)(ab)=a(ba)b=a(ab)b=(aa)(bb)=a²b²